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Transition from regular to complex behaviour in a discrete deterministic asymmetric neural network model

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Abstract. We study the long-time behaviour of the transient before the collapse on the periodic attractors of a discrete deterministic asymmetric neural network model. The system has a finite number of possible states so it is not possible to use the term chaos in the usual sense of sensitive dependence on the initial condition. Nevertheless, on varying the asymmetry parameter, k , one observes a transition from ordered motion (i.e. short transients and short periods on the attractors) to 'complex' temporal behaviour. This transition takes place for the same value k_c at which one has a change in behaviour for the mean transient length from a power law in the size of the system (N) to an exponential law in N . The 'complex' behaviour during the transient shows strong analogies with chaotic behaviour: decay of temporal correlations, positive Shannon entropy, non-constant Renyi entropies of different orders. Moreover the transition is very similar to the intermittent transition in chaotic systems: a scaling law for the Shannon entropy and strong fluctuations of the 'effective Shannon entropy' along the transient, for $k > k_c$.

1. Introduction

The properties of the low-temperature phase of spin systems with random quenched couplings have been the object of intensive analysis in recent years (Mézard *et al* 1987, Fischer and Hertz 1991). The interest in such models comes from the study of both spin glasses and neural networks (Amit 1989). One of the best known among these models, the Serrington–Kirkpatrick (SK) model, is a fully connected version of a spin glass. Here the interactions between pairs of spins are random, quenched and symmetric. This leads to a large number of metastable states which are responsible for the complicated dynamic behaviour of this and related models.

From a biological point of view the constraint of symmetric interactions is unsatisfactory. In fact, in the language of neural networks, two neurons will not act on each other in a symmetric way. It is then natural to investigate models with asymmetric couplings between spins (Amit 1989).

Apart from these biological motivations, the study of disordered spin systems with asymmetric bonds is also an interesting problem from the viewpoint of non-equilibrium statistical mechanics. In fact, for these models one no longer has a Hamiltonian and a detailed balance condition, and therefore no fluctuation dissipation theorem exists (Crisanti and Sompolinsky 1987). This last relation between the response and the correlation functions is, however, essential for extracting the long-time properties of the dynamics. As a consequence, the long-time limit has to be calculated via the full dynamic problem and cannot be evaluated by equilibrium statistical mechanical averages.

Up until now such models have been principally studied in the framework of neural network model, and the interest has been mainly focused on the properties of the attractors of networks of spins with asymmetric couplings. In particular the effect of the degree of asymmetry on the long-time behaviour has been analysed. Although analytical results, based on perturbation expansions in the nonlinearity parameter of soft spin models (Crisanti and Sompolinsky 1987), showed a dramatic change in the deterministic dynamics as soon as the asymmetry was switched on, subsequent numerical data are in agreement with a transition at a finite degree of asymmetry (Spitzner and Kinzel 1989). A perturbation expansion in the asymmetry parameter for the hard Ising spin seems to suggest a transition at a finite value of the asymmetry parameter (Rieger et al 1989). The location of the transition is, however, controversial.

An interesting feature of these models is the presence of extremely long transients (Crisanti and Sompolinsky 1988, Nützel 1991), i.e. the time the system wanders around before it settles down in the 'asymptotic' regime. It has been found that, while for nearly-symmetric networks, the typical number of updates before the relaxation onto the attractor grows as a power of the system size, for sufficiently strong asymmetry the transients are typically exponential in the size of the system (Nützel 1991). Therefore it is natural to study the properties of the transient, because they give the features of a sufficiently large system during any reasonable time.

In this paper we investigate the deterministic parallel dynamics of a fully connected Ising spin system with random quenched asymmetric interactions. We stress that if the spins are continuous, then this system exhibits chaotic behaviour, e.g. positive Lyapunov exponent, in the limit of high asymmetry (Sompolinsky et al 1988, Tirozzi and Tsodyks 1991). In the case of Ising spins, however, because of the discrete nature of the states, it is not possible to characterize the time behaviour of the system by a straightforward application of the most simple methods of chaotic dynamics, e.g. by the computation of the Lyapunov exponents. Nevertheless one can perform a study of the temporal evolution in terms of well defined quantities, such as Shannon and Renyi entropies, which give a non-ambiguous measure of the 'complexity' of the system. From these indicators and from the behaviour of the correlation function one has strong evidence for a transition on varying the asymmetry parameter from regular to 'complex' behaviour in systems of finite size. This transition is very similar to the onset of turbulence in the intermittent scenario in chaotic dynamic systems (Pomeau and Manneville 1980).

The model will be described in section 2; section 3 deals with the chaotic—i.e. complex, in the information theory meaning—behaviour during the transient and in section 4 we study the fluctuations of the chaotic degree along the trajectories, in terms of the Renyi entropies. In section 5 the reader will find a summary and conclusions.

2. The model

In this paper we consider an asymmetric spin glass model with deterministic dynamics (Crisanti and Sompolinsky 1987). The model consists of N fully connected Ising spin $\sigma = \pm 1$ interacting via quenched random asymmetric couplings of the form

$$J_{ij} = J_{ij}^S + k J_{ij}^A \quad k \geq 0 \quad (2.1)$$

where $J_{ji}^S = J_{ij}^S$ and $J_{ji}^A = -J_{ij}^A$, and (i, j) denotes a pair of spins. The diagonal elements J_{ii}^S are zero. The elements of the upper triangular part of the symmetric and antisymmetric

matrices \mathbf{J}^S and \mathbf{J}^A are random independent Gaussian variables with zero mean and mean square equal to $1/(N-1)(1+k^2)$ so that the rows of the couplings matrices have average norm 1 independently of the system size or symmetry.

The parameter k defines the symmetry of the coupling, and ranges from $k = 0$ for symmetric through $k = 1$ for asymmetric to $k = \infty$ for antisymmetric couplings. In the literature the related parameter λ defined by

$$\lambda = \frac{1 - k^2}{1 + k^2} = N[J_{ij} J_{ji}] \tag{2.2}$$

ranging from $\lambda = 1$ for symmetric through $\lambda = 0$ for asymmetric to $\lambda = -1$ for antisymmetric matrices is often used. In (2.2) square brackets denote the average over the realization of the coupling matrix \mathbf{J} .

Our analysis is restricted to deterministic parallel dynamics, i.e. the spins are updated simultaneously according to

$$\sigma_i(t + 1) = \text{sgn} \left[\sum_{j=1}^N J_{ij} \sigma_j(t) \right]. \tag{2.3}$$

For any given sample \mathbf{J} the dynamics is deterministic and depends only on $S(0)$ the spin configuration $\{\sigma_i\}$ at time $t = 0$.

For a system of N spins there are 2^N different possible states. The evolution (2.3) connects these states. Therefore for each realization of the couplings the structure of the dynamics can be depicted in terms of oriented graphs, i.e. a set of points representing the states connected by arrows indicating the transitions. An example is shown in figure 1. Clearly each state has one and only one outgoing arrow, but different arrows can end at the same state. Since the number of states is finite and the dynamics is deterministic, each initial configuration of spins evolves to a definite attractor, which can be either a fixed point $S(t + 1) = S(t)$ or a periodic repetition of l configurations $S(t + l) = S(t)$ (cycle of length l).

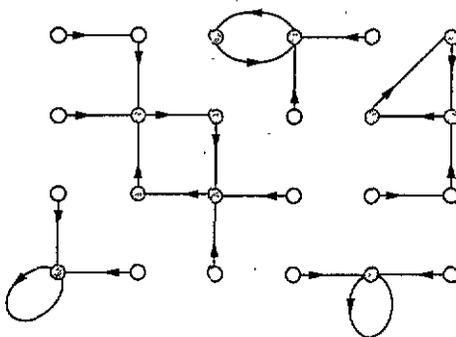


Figure 1. Some examples of possible behaviour. One recognizes two fixed points and three cycles of period 2, 3 and 4, respectively.

From this remark one may be tempted to conclude that the behaviour of (2.3) is 'trivial' in the sense that there is no possibility of having behaviour similar to chaotic behaviour.

This is not completely true. In fact (2.3) can be seen as a cellular automaton for which the behaviour just described is common. Nevertheless, it is well known that there is a wide class of cellular automata with 'complex' behaviours.

The reason is that the time it takes to reach the attractor can be very long. As a consequence for a long time the system may exhibit a complex behaviour before it eventually relaxes to the 'asymptotic' regime, usually a cycle. The presence of long transients, exponential in the system size, was first observed for sequential dynamics (Crisanti and Sompolinsky 1988) and recently analysed in more detail for both sequential and parallel dynamics (Nützel 1991).

From the above considerations it follows that in these models there are two relevant time scales: the relaxation time τ and the cycle length l . These are defined as

$$l = \min_n [S(t+n) = S(t)] \quad (2.4)$$

$$\tau = \min_n [S(l+n) = S(n)]. \quad (2.5)$$

The time τ is the time the system needs to reach the first cycle of length l . In general both l and τ depend on the initial state $S(0)$ and on the coupling realization J_{ij} . For any given k and system size N one can define the average transient time and the average cycle length. To this end one considers a large number M of independent coupling realizations J_{ij} . For each sample and a random initial state $S(0)$ the time τ and the cycle length l are recorded. One then defines the average values as

$$\langle l \rangle = \frac{1}{M} \sum_{m=1}^M l(m) \quad (2.6)$$

$$\langle \tau \rangle = \frac{1}{M} \sum_{m=1}^M \tau(m) \quad (2.7)$$

where $l(m)$ and $\tau(m)$ are the values of l and τ defined in (2.4) and (2.5) for the m th sample.

Studying these average quantities as a function of both the symmetry and the system size, Nützel found strong numerical evidence for a transition between two different behaviours as the symmetry of the couplings is varied. He found that for highly symmetric couplings $\langle l \rangle$ is size independent and $\langle \tau \rangle$ grows as a power of N , while the behaviour of both quantities is exponential in N for highly asymmetric couplings. He located the transition point around $\lambda \simeq 0.5$ where $\langle l \rangle$ shows strong changes (Nützel 1991). By contrast $\langle \tau \rangle$ changes smoothly with λ . This behaviour was found for both sequential and parallel dynamics.

The values of l and τ , however, exhibit strong sample-to-sample fluctuations. As a consequence the leading contributions to the average (2.6) and (2.7) come from rare events. This suggests that the appropriate quantity for averaging is not τ and l , but their logarithms. One then defines the 'typical' values

$$\tau_{\text{typ}} = \exp(\ln \tau) \quad (2.8)$$

$$l_{\text{typ}} = \exp(\ln l). \quad (2.9)$$

An analysis of these quantities as a function of both the symmetry of the couplings and the system size reveals that

$$\tau_{\text{typ}} \sim \langle \tau \rangle \propto N^{\alpha(k)} e^{\beta(k)N} \quad (2.10)$$

where $\beta(k) \ll 1$ for $k_c \approx 0.5$. Similar behaviour is observed for the cycle length l .

We find that, for $k \geq k_c$, τ has a roughly log-normal distribution whose peak and width scale linearly with N . In other words $\ln \tau$ is a Gaussian with

$$\langle \ln \tau \rangle \propto N \tag{2.11a}$$

$$\langle (\ln \tau)^2 \rangle - \langle \ln \tau \rangle^2 \propto N^2. \tag{2.11b}$$

The same result also remains true for sequential dynamics (Crisanti and Sompolinsky 1988). The log-normal distribution is quite common in disordered systems (Paladin and Vulpiani 1987). However, in these models one has non-standard scaling of the variance of $\ln \tau$. In fact usually both the mean value and the variance scale with N . This means that here there are enormous fluctuations: the ratio between the standard deviation and the mean value is constant instead of being proportional to $N^{-1/2}$ as in the standard case.

The presence of these long transients, e.g. for $k = 0.8$ and $N = 100$ one has $\tau_{\text{typ}} = O(10^6)$, makes it natural to study the dynamics during the transient. Not only because this is the behaviour of the system during reasonable times, but also because this is the behaviour which is revealed by any mean-field study of these systems.

To analyse the temporal evolution one of the first quantities one usually studies is the time correlation. For a given initial state $S(0)$ and coupling realization J_{ij} we compute the following quantity

$$C(t) = \frac{1}{N} \sum_{i=1}^N \overline{\sigma_i(t+t') \sigma_i(t')} \tag{2.12}$$

where the bar denotes time average. The correlation $C(t)$ is evaluated taking into account only the data for the transient, i.e. times less than τ . We remark that $C(t)$ is not averaged over different realizations of J_{ij} . From a comparison of several realizations and different N , it seems that $C(t)$ is a self-averaging quantity that depends very weakly on N . In contrast $c_i(t) = \overline{\sigma_i(t+t') \sigma_i(t')}$ changes at varying i and there exist two different possibilities: $c_i(t)$ either decreases monotonically with t ; or it shows strong oscillations with a decreasing envelope.

For $k = 1$ (full asymmetric case) one has $C(t) = 0$ for $t \geq 1$. The behaviour of $C(t)$ for $k < 1$ is reported in figure 2(a). Unfortunately there is no simple formula to fit $C(t)$ as a function of t at different k , even if, for each k , the data seem consistent with a stretched exponential decay:

$$C(t) \sim \exp[-a t^b] \quad \text{with } b < 1. \tag{2.13}$$

Nevertheless one can define a characteristic time, τ_c , as the time at which the envelope of $C(t)$ reaches a threshold value, e.g. $1/e$. Figure 2(b) shows $1/\tau_c$ against k ; there is evidence for a transition at $k \approx 0.5$. The above scenario does not change even when using different values for the threshold.

3. Chaotic behaviour during the transient

It is known (Isola *et al* 1985) that in some dynamical systems with many degrees of freedom the transition from regular-to-chaotic motion is also signalled by a change, from a power to an exponential in the number of degrees of freedom, of the Poincaré return time.

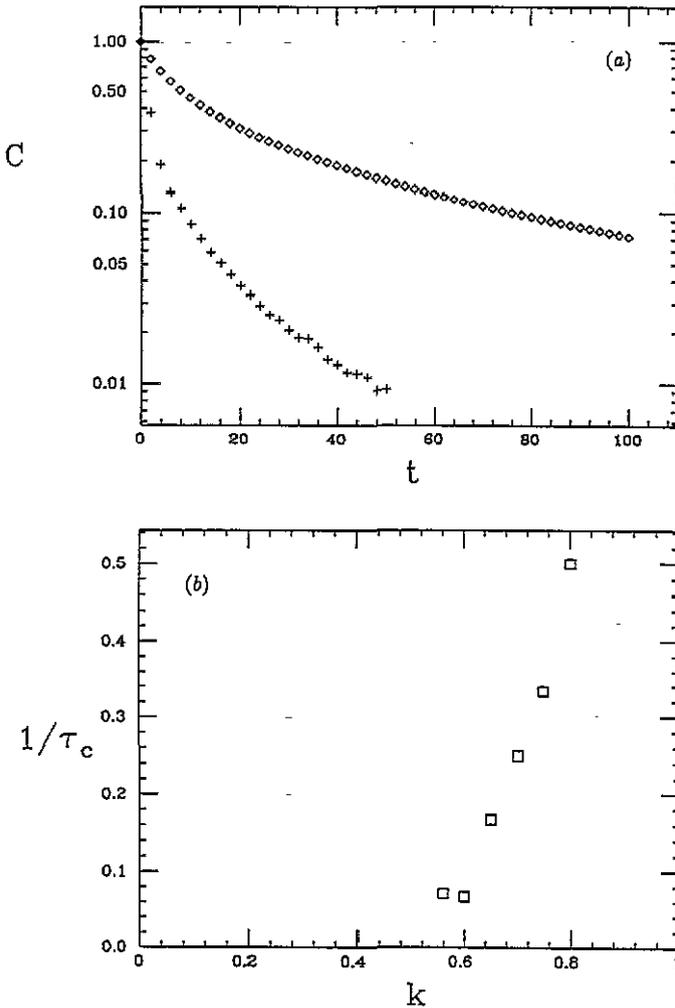


Figure 2. (a) $C(t)$ as a function of t for $k = 0.6$ (diamonds) and $k = 0.8$ (plus); in order to demonstrate the main feature we plot $C(t)$ only at even values of t . (b) $1/\tau_c$ as a function of k . The system size is $N = 200$.

This result and the analysis of the correlation functions discussed in the previous section suggest the possibility of 'chaotic behaviour' during the transients for sufficient asymmetric couplings. This is in spite of the fact that a stationary periodic orbit is always found to be the final attractor. The reason to study this chaotic behaviour in more detail follows from the fact that if the asymmetry is strong enough the transient increases exponentially with the system size. Therefore any practical investigation of the time evolution is necessarily limited to the transient. Moreover one can argue that the attractor might not be relevant for the dynamics since even in the presence of stable solutions, generic initial conditions relax towards it only after exponentially long transients (Crutchfield and Kaneko 1988).

Due to the discrete nature of the spin and to the full connectivity of the system a quantification of 'chaos' during the transient is not at all straightforward. In fact, for systems whose state changes continuously one can quantify the degree of chaos in terms of the Lyapunov exponents (Benettin *et al* 1980a, b). They measure the exponential growth

rate of the distance between two initially close trajectories.

One can try to follow the idea of Lyapunov exponents by studying two trajectories whose initial condition $S(0)$ and $S'(0)$ differ by a single spin. Looking at the growth of the Hamming distance between $S(t)$ and $S'(t)$ one can then introduce something similar to the first Lyapunov exponent. This is the so called 'damage spreading' method which proved to be useful for discrete systems similar to (2.3) but with only local interactions among the spins (Kauffman 1969, Derrida and Stauffer 1986).

In this paper we use a completely different approach to characterize the chaotic behaviour. Following the basic idea of Shannon in information theory (Khinchin 1957), we look at the 'complexity' of the temporal history on the i th spin during the transient (Grassberger 1986). For each initial configuration and fixed realization of J_{ij} the 'life history' of a spin σ_i before the system relaxes to the attractor can be written as a sequence

$$\sigma_i(0), \sigma_i(1), \dots, \sigma_i(m) \tag{3.1}$$

where $m < \tau + l$. Since the transient increases exponentially with system size, in principle the length of the sequence (3.1) can be as large as we want by increasing N .

The complexity of the temporal behaviour of the spin can be characterized by looking at the average amount of information contained in the sequence (3.1). To this end we consider a sequence ω_n of n possible outcomes of the spin σ_i . Since it can assume only two values there are 2^n possible sequences. These can be regarded as strings of n binary digits. In general, the sequences ω_n will appear in the life history of the spin σ_i with different frequencies. Some of them may not appear at all or with very small probability. By computing the frequency with which each ω_n appears in the sequence (3.1) we can define the probability $P(\omega_n)$ that the dynamics generates the sequence ω_n . The measure of information contained in $P(\omega_n)$ is given by the entropy $H(n)$ defined as

$$H(n) = - \sum_{\omega_n} P(\omega_n) \ln P(\omega_n) \tag{3.2}$$

where the sum is over all sequences ω_n . If sequence (3.1) is stationary the quantity $H(n)$ depends only on n and not on the initial state $\sigma_i(0)$. In this case, in the limit of large n the average information $H(n)/n$ converges to the limit value h which is called the Shannon entropy of sequence (3.1) (Khinchin 1957). The existence of this limit implies that, for large n , the mean gain of information in considering sequences of length $n + 1$ instead of n , i.e.

$$h(n) = H(n + 1) - H(n) \tag{3.3}$$

approaches a constant limit given by the Shannon entropy h .

If the sequence (3.1) is a Markov chain, then $h(n) = h$ for each n . This can be generalized to Markov chains of order p , i.e. the probability of having an event at time t depends only on the events at times $t - p$ up to $t - 1$, in which case $h(n) = h$ for $n \geq p$.

The relevance of h follows from the first Shannon-McMillan theorem (Khinchin 1957) which states that for large n the number of n -term sequences ω_n one can really observe in (3.1) is

$$N_{\text{eff}}(n) \sim e^{hn} \tag{3.4}$$

This can also be stated as follows. For n sufficiently large all the n -term sequences ω_n can be separated into two classes $\Omega_1(n)$ and $\Omega_2(n)$, such that for every sequence in the first class $P(\omega_n) \simeq \exp(-hn)$ and

$$\sum_{\omega_n \in \Omega_1(n)} P(\omega_n) \rightarrow 1 \quad \text{for } n \rightarrow \infty.$$

In contrast, for the sequences in the second class

$$\sum_{\omega_n \in \Omega_2(n)} P(\omega_n) \rightarrow 0 \quad \text{for } n \rightarrow \infty.$$

If we remember that the maximum number of n -term sequences is

$$N_{\max}(n) = 2^n = e^{n \ln 2}$$

we see that if $h < \ln 2$ for large n the class $\Omega_1(n)$ contains only a negligible fraction of sequences. In contrast, the overwhelming majority of such sequences fall into $\Omega_2(n)$.

If the probability of having $\sigma_i = 1$ or $\sigma_i = -1$ in (3.1) is the same, as it is in our case, then $h < \ln 2$ signals the presence of a rule in the sequence. Thus the Shannon entropy gives a measure of the 'complexity' of the sequence. In particular if $h = 0$ then we are in the presence of periodic motion, while if $h = \ln 2$ the sequence is generated by a head-and-tail trial. We stress that $\ln 2$, the largest value attainable by h , is reached only if the symbols in (3.1) are independent and equiprobable. In fact $h < \ln 2$ also if the symbols are independent but not equiprobable, e.g. generated according to a head-and-tail trial done with an unbalanced coin (Khinchin 1957).

To analyse the motion of the system during the transient, we have computed the Shannon entropy of different spins for a fixed, randomly chosen configuration of couplings J_{ij} . In all cases we have taken N large enough to ensure sufficiently long transients: $\tau \sim O(10^5)$ at least. For $k \geq 0.7$ we considered $N = 100$, $N = 200$ and $N = 300$, while for $k < 0.7$ we used $N = 200$ and $N = 300$. For smaller value of k the required system size becomes very large and, since the computational time increases as N^2 , it is practically impossible to reach values of k which are too small. In all the cases we checked that the sequence (3.1) is stationary for each spin.

By computing $h(n)$ for different n we found a rapid convergence towards the Shannon entropy h . In figure 3 a typical case from which we see that $n = 4$ or $n = 5$ is sufficient for a good estimate of h is shown.

In the limit case of $k = 1$ the evaluated h is very close to the largest value $\ln 2$. This is not surprising since if the couplings J_{ij} and J_{ji} are uncorrelated then the statistical properties of the dynamical system (2.3) are the same as the sequence generated by a head-and-tail trial (Gutfreund et al 1988).

For a given configuration of couplings J_{ij} and a fixed value of N we have computed the Shannon entropy $h^{(i)}$ for each sequence (3.1). The mean value

$$\langle h \rangle = \frac{1}{N} \sum_{i=1}^N h^{(i)}$$

seems to be a self-averaging quantity, which does not depend on N ; moreover the variance

$$\frac{1}{N} \sum_{i=1}^N (h^{(i)} - \langle h \rangle)^2$$

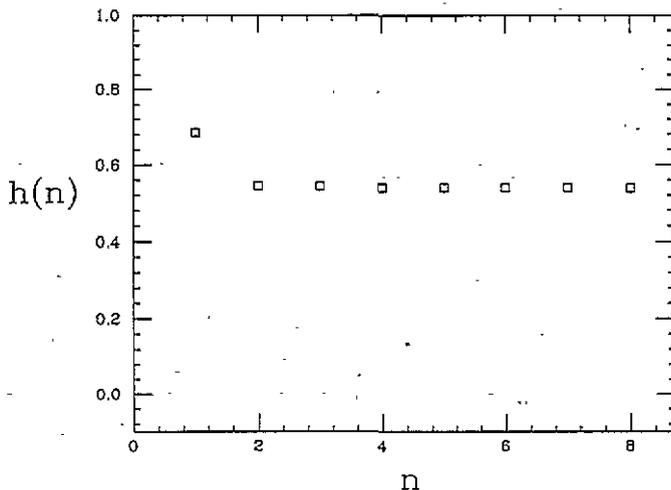


Figure 3. $h(n)$ as a function of n for one spin sequence at $k = 0.7$ and $N = 200$.

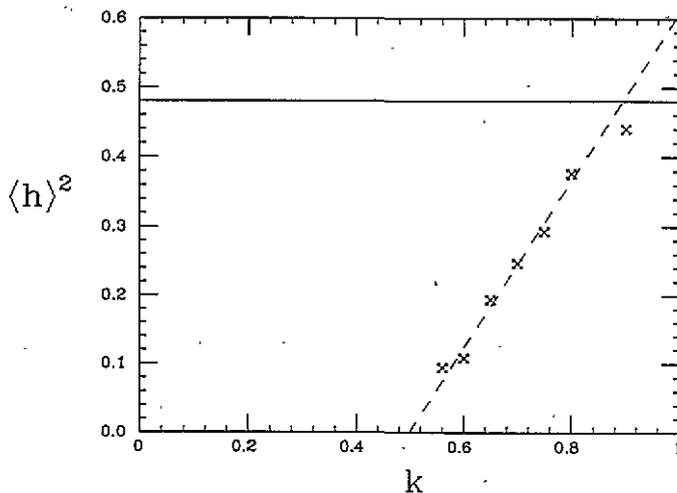


Figure 4. $\langle h \rangle^2$ as a function of k for $N = 200$; the broken line indicates the law $\langle h \rangle \propto (k - k_c)^{1/2}$ with $k_c = 0.50$. The horizontal line corresponds to $\langle h \rangle = \ln 2$.

decreases for increasing N . In figure 4 we show $\langle h \rangle^2$ as a function of k : it is well evident that the scaling

$$\langle h \rangle \sim (k - k_c)^{1/2} \quad \text{with } k_c = 0.50. \tag{3.5}$$

We close this section with the following remark. Note that the point at $k = 0.56$ does not agree very well with equation (3.5). Since it is practically impossible for us to work with smaller values of k , we cannot say whether this point can be considered as evidence for crossover behaviour. In section 5 we shall reconsider this issue.

4. Fluctuations during the transient

In general not all the 2^n possible n -term sequences will appear in sequence (3.1) generated by the dynamics. Nevertheless, for large n their number $\tilde{N}(n)$ increases exponentially with n . The rate of growth

$$K_0 = \lim_{n \rightarrow \infty} \frac{1}{n} \ln \tilde{N}(n) \quad (4.1)$$

is the topological entropy. The smaller K_0 is, the larger the number of forbidden n -term sequences in (3.1). For our system we found that at any value of k the topological entropy assumes its maximum value $K_0 = \ln 2$. This means that for fixed J_{ij} and a generic initial condition, all possible sequences are actually generated by (2.3).

The topological and Shannon entropies are global quantities, since they ignore the finite-time, i.e. finite n , fluctuations. For any finite n we can define the 'effective Shannon entropy' $\gamma(n)$ of the n -term string ω_n as

$$\gamma(n) = -\frac{1}{n} \ln P(\omega_n). \quad (4.2)$$

In general the value of $\gamma(n)$ will depend on the string ω_n , even for large but finite- n . These finite- n fluctuations can be characterized by means of the Renyi entropy (Renyi 1970, Paladin and Vulpiani 1987). Let us define

$$H_q(n) = -\frac{1}{q-1} \ln \left[\sum_{\omega_n} P(\omega_n)^q \right] \quad (4.3)$$

where the sum is extended over all the 2^n n -term sequences ω_n . In the limit of large n the quantity

$$h_q(n) = H_q(n+1) - H_q(n) \quad (4.4)$$

or, equivalently $H_q(n)/n$, converges towards the limit h_q which is called the Renyi entropy of order q .

From the definition it readily follows that the Shannon and topological entropies are obtained from h_q via $h = \lim_{q \rightarrow 1} h_q$ and $K_0 = \lim_{q \rightarrow 0} h_q$. It is possible to show that, in general, h_q is a non-increasing function of q (Renyi 1970). In the limit case of absence of fluctuations one has $h_q = K_0 = \text{constant}$. Thus deviation of h_q from K_0 gives a measure of the relevance of the fluctuations.

As in section 3 for the Shannon entropy, we have computed the Renyi entropies $h_q^{(i)}$ for each sequence $\sigma_i(1), \sigma_i(2), \dots$. For values of k close to 1 $h_q^{(i)}$ does not change very much on varying q , while for k close to k_c one observes large deviations of $h_q^{(i)}$ from $h^{(i)}$. In figure 5 we show $\langle h_q \rangle$ as a function of q for $k = 0.9$ and $k = 0.7$: even at a qualitative level, the difference is well evident. The properties discussed in section 3, for the Shannon entropy, also hold true for h_q : each $\langle h_q \rangle$ is a self-averaging quantity which does not depend on N .

The Renyi entropies are closely related to the probability distribution $P_n(\gamma)$ of the 'effective Shannon entropy' $\gamma(n)$. For large n it is reasonable to make the ansatz that $P_n(\gamma)$ vanishes exponentially (Eckmann and Procaccia 1986, Paladin et al 1986),

$$P_n(\gamma) \propto \exp[-nS(\gamma)] \quad S(\gamma) \geq 0. \quad (4.5)$$

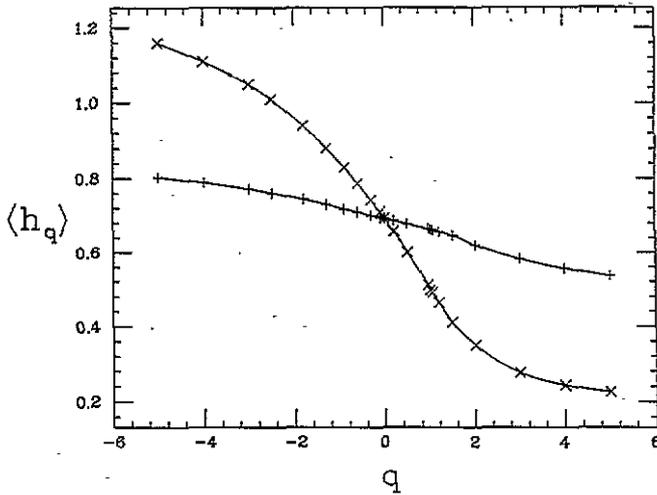


Figure 5. $\langle h_q \rangle$ as a function of q for $k = 0.9$ (plus) and $k = 0.7$ (cross). The system size is $N = 200$.

The equal sign holds only for $\gamma = h$, as a consequence of the Shannon–McMillan theorem. With this assumption one finds

$$h_q = \frac{1}{q - 1} \min[\gamma(q - 1) + S(\gamma)]. \tag{4.6}$$

This is a Legendre transform which shows that each q selects a particular class of sequences with ‘effective Shannon entropy’ γ . In particular $q = 1$ selects the most probable sequences, whereas for $q \neq 1$ the leading contribution to h_q comes from rare sequences.

For $\gamma \simeq h$ one has the following parabolic approximation for $S(\gamma)$

$$S(\gamma) = (\gamma - h)^2 / 2\mu \tag{4.7}$$

where μ is given by

$$\begin{aligned} \mu &= \lim_{n \rightarrow \infty} \frac{1}{n} \left[\sum_{\omega_n} P(\omega_n) [\ln P(\omega_n)]^2 - \left[\sum_{\omega_n} P(\omega_n) \ln P(\omega_n) \right]^2 \right] \\ &= \lim_{n \rightarrow \infty} n \int (\gamma - h)^2 P_n(\gamma) d\mu. \end{aligned}$$

Equation (4.7) corresponds to

$$h_q = h - \mu(q - 1)/2 \quad q \simeq 1. \tag{4.8}$$

One can consider μ as an inhomogeneity factor. The (4.7) and (4.8) correspond to a Gaussian approximation. The quantities h and μ are the most relevant characterizations for the ‘complexity’ of the sequence since h gives the typical value and μ is related to the variance of the fluctuations.

We found that, for k close to 1, μ assumes very small values, indicating that the sequences are almost homogeneous. When k decreases μ increases and assumes its largest

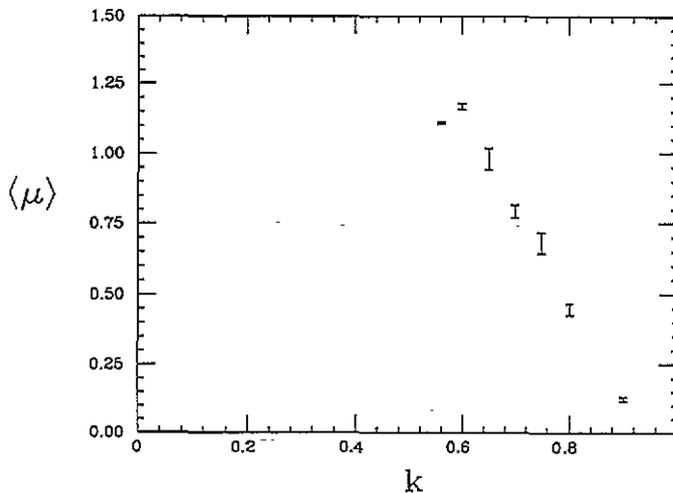


Figure 6. $\langle \mu \rangle$ as a function of k for $N = 200$.

values (strong fluctuations) as one reaches the critical value k_c . In figure 6 we show $\langle \mu \rangle$ as a function of k .

It is interesting to note that the behaviour of $\langle h \rangle$ and $\langle \mu \rangle$ as function of the parameter k is similar to that observed for the intermittent transition in dynamical system, e.g. in the Lorenz model (Pomeau and Manneville 1980).

We close this section by noting that all the results for $\langle h \rangle$ and $\langle \mu \rangle$ for k larger than approximately 0.5 are practically independent of N in the range $N = 50 \div 500$.

5. Summary and conclusions

In this paper we have studied the long-time behaviour of binary sequences, obtained from the deterministic evolution of an asymmetric neural network model with N components. The system size N is large but finite. In spite of the discrete nature of its states, on varying the asymmetry parameter k , one observes some features which are very similar to intermittent transition in chaotic systems. For $k < k_c$ one has that the typical length of the transient follows a power law in the size of the system N , while for $k > k_c$ the transients increase exponentially with N . One can define, in an unambiguous way, some degree of 'complexity' during the transient, by using a well established concept (Shannon and Renyi entropies) in information theory.

For $k > k_c$ the time behaviour of the transient is very similar to that of chaotic systems: decay of the temporal correlation and positive Shannon entropy. Around k_c one has a scaling law for the mean Shannon entropy $\langle h \rangle$:

$$\langle h \rangle \sim \sqrt{k - k_c}$$

moreover the fluctuations of the degree of chaos, given by the mean inhomogeneity factor $\langle \mu \rangle$, are large for k approaching k_c^+ .

All our results on Shannon and Renyi entropies and characteristic times for k larger than approximately 0.5 do not depend on the value of N in the investigated range $N = 50 \div 500$.

However, in principle, we cannot exclude the possibility that what we have found is not a real sharp transition, but the manifestation, at finite N , of crossover behaviour. For instance, for sequential dynamics (Crisanti and Sompolinsky 1988) one has exponential growth with N for τ_{typ} according to (2.10) even for $k < k_c$, but the constant $\beta(k)$ is so small that in order to see the exponential law one has to consider enormous systems, $N = O(10^3)$. In a way similar to that described in Crisanti and Sompolinsky (1988) the average correlation function during the transient has a characteristic time τ_c proportional to k^{-6} for small k .

From our computations and a previous study by Nützel (1991) one can conclude that the dependence of k_c on N , if any, is very weak. For almost all practical purposes, however, the nature of the transition, i.e. sharp or crossover, is not very important. Nevertheless, its existence is relevant for numerical studies. For example, it could be very dangerous to compute temporal averages at different k using the same N and maximal time. Indeed the meaning of these averages is completely different: for $k > k_c$ one should have information about the transient while for $k < k_c$ one just observes the properties of periodic attractors.

We conclude by noting that similar problems are also present in Hamiltonian (and more generally symplectic) systems with many degrees of freedom (Falcioni *et al* 1991).

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